Why We Fight Online Appendix
Proofs of illustrative examples

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This document provides formal foundations for the pie-splitting examples in *Why We Fight*. There are a number of ways to formalize these models, and in general we have opted for the simplest illustrations that remain close to the specific example presented in the book.

1 Chapter 1: The incentives for peace

Pachelly and Los Chatas

Setup

Imagine Bello, on the northern edge of Medellin, is worth (in present value) $100 in extortion, drug corners, and other criminal rents. And suppose that the two criminal groups, Pachelly and Los Chatas, have equal shares of the neighborhood (valued in $50 each). Let's also say that, militarily-speaking, both groups are evenly matched. This means, if it comes to war, each gang has an equal chance of winning (50 percent). Assume the gang that wins the fight gets the whole of Bello forever; the loser gets nothing. Both groups know the move of the other player, the payoffs, and their probability of winning. The two rivals know, as we do, that war has dire consequences—killing soldiers, destroying business, and attracting police attention. In the struggle for Bello, let's suppose that both gangs expect fighting to destroy a fifth of the pie—$20.

Notation

- Subscript i denotes each of the two rival groups—Pachelly (P) and Los Chatas (C)—while j is just the notation assigned to the rival of i
- $U_i(\cdot)$ is the utility group i gets from taking an action (either choosing *war* or *peace*)
- R is a generic way to denote the total value/size of the pie ($100 in our case)
- p_i is the probability of victory by side i (0.5 in this example)
- c_i is the cost of war for group i ($20 in this example)
- x_i is the share of the pie either side receives
- Finally, we will denote s_i a pure strategy of group i, which can be either war or peace.[1]
- $E[U_i(s_i, s_j)]$ will denote the (expected) payoff of group i of choosing strategy s_i given that the other player chooses strategy s_j. It measures the expected value they will have for a particular strategy profile. We will denote (s_P, s_C) a strategy profile, where the first action is the one of Pachelly and the second one of Los Chatas.

[1] s_i could also be a probability distribution over war and peace (e.g. play with 60% of probability war and 40% of probability peace) known as mixed strategies. We will focus on equilibrium in pure strategies so we won't consider this kind of strategies.

Claims in the book

1. **Given the initial distribution, Tom would happily choose peace. Pachelly faces identical incentives and will also be happier with 50% of the pie over war.**
2. **Imagine that Tom's lieutenants outmaneuvered Pachelly and occupied a series of drug corners without a shot fired. Should Pachelly launch an attack to get back their plazas?** ***So long as Los Chatas controlled at most $60 of Bello, Pachelly won't have an incentive to choose war.***
3. **What happens if the balance of military power shifts in Medellín? Imagine that Pachelly makes an honest assessment of its new strength and figures its chance of winning has fallen from 50 to 20 percent. Now Tom and Los Chatas could now win a war 80 percent of the time. Nothing else has changed so far, and so Tom and Los Chatas still occupy half of Bello.**

   Regardless, ***Tom will not launch a war and Pachelly will prefer any peaceful deal that leaves them at least with 16% of Bello.*** **The bargaining range where both sides has shifted, from one where Pachelly is content with $40 to $60, to one where they'll grudgingly accept $16 to $36.**

Proofs

1. Each group is going to compare the expected value of peace to the expected value of war to decide on their best response.[2]

$$
\begin{aligned}
E[U_i(peace, peace)] &> E[U_i(war, peace)] \quad \text{for } i = \text{P,C} \\
x_i &> p_i \times (R - c) + (1 - p_i) \times 0 \\
x_i &> p_i \times (R - c) \\
x_i &> 0.5 \times 80 \\
x_i &> 40
\end{aligned}
$$

   If a gang starts with x_i=50 then it's their best response to choose peace given that the other gang chooses peace. Each gang also grasps that a sure bet of $50 is better than a risky shot at $40, so both play peace and it is a Nash equilibrium.

   A Nash equilibrium is a strategy profile that for each player their action is the best response given other players' actions. Then if originally they are at peace they have no incentives to choose war.

   This implies a bargaining range where $x_i \in (40, 60)$ for both parties.

2. For peace to be sustainable we need for peace is a best response for Pachelly after the seizure. This requires that:

$$
\begin{aligned}
E[U_P(peace, peace)] &\geq E[U_P(war, peace)] \\
x_P &\geq 40
\end{aligned}
$$

   Suppose that Los Chatas seizes t territory. Peace is Pachelly's optimal best move if:

[2] A best response is an action that provides a payoff higher or equal to any other possible action given the action of the other group. Formally, s_i is a best response to s_j if $u_i(s_i, s_j) \geq u_i(s_i', s_j)$ where s_i' represent any other strategy that could be chosen by player i.

$$
\begin{aligned}
E[U_P(peace, peace)] &\geq E[U_P(war, peace)] \\
50 + t &\geq 40 \\
t &\leq 10
\end{aligned}
$$

The seizure doesn't put the rivals outside their equilibrium bargaining range. A cool-headed, calculating Pachelly will resign themselves to the new events as long as they control at least 40% of Bello. Clearly for Los Chatas is beneficial to choose peace as they have a higher share and the expected value of war remain constant, which means that (peace,peace) remains a Nash equilibrium.

3. Now p_P has fallen from 0.5 to 0.2, and (1-p_P)=p_C has risen from 0.5 to 0.8. The bargaining range shifts accordingly, and Pachelly may need to make transfers to Los Chatas that will make peace sustainable (remain a Nash equilibrium). First, we need that the transfer is such as peace remains a best response for Los Chatas (given that Pachelly chooses peace):

$$
\begin{aligned}
E[U_C(peace, peace)] &\geq E[U_C(war, peace)] \\
x_{P,initial} + t &\geq p_C \times (R - c) \\
50 + t &\geq 0.8 \times 80 \\
t &\geq 14
\end{aligned}
$$

Additionally, we need the same condition for Pachelly:

$$
\begin{aligned}
E[U_P(peace, peace)] &\geq E[U_P(war, peace)] \\
x_{C,initial} - t &\geq p_P \times (R - c) \\
50 - t &\geq 0.2 \times 80 \\
34 &\geq t
\end{aligned}
$$

Hence, $t \in [14, 34]$ is the range for the transfer that Pachelly needs to make to Los Chatas that makes peace sustainable after the power shift. The bargaining range has shifted to $x_C \in [66, 86]$.

2 Chapter 2: Unchecked private interests

Colonists and the Crown

Setup

Imagine that all the 13 colonies' land, taxes, and other spoils are a pie that the British Crown and the American colonists must share between them. Now it was time for the colonists to pay their share of their continent's defense and administration. And so, the Crown began to levy taxes.

The Colonists were outraged. Why should they pay more, when they haven't a formal say in this faraway government? But then, why didn't the Crown grant the Colonists representation in exchange for more taxes? Revolution is puzzling because fighting would be long and brutal. These costs shrink the pie in $20 after war happens. Assuming leaders weigh all the costs and benefits, both sides have more to profit from a peaceful split of the $100 colonial pie. The key phrase here is "assuming leaders weigh all the costs and benefits". Economists call this the "unitary actor" assumption. It means the ruler is trying to maximize the group's collective interests, not their own.

Assume the colonists and the crown have equal power, that is, they are equally likely to win a war, and both actors know it.

Notation

Notation is as before, with the following differences:

- Subscript i denotes each of the two rivals: the British Crown (B) and George Washington (W). Washington decides on behalf of all Colonists (C) in this example.
- $\alpha \in [0, 1)$ is the share of the colonial side's costs that Washington bears.
- b_p is the share of x_C that Washington gets under peace, and b_w is his share if there is war and the colonists win

Claims in the book

1. **Assume Washington and the Crown leaders weigh all the costs and benefits of war. *Then both parts will accept a peaceful split offering at least 40% of the pie.***
2. **Dispense now the unitary actor assumption and let Washington alone choose whether the colonies go to war against the crown. Additionally, suppose now that Washington bears only part of the cost. *Then, peace is still predicted but the bargaining range is reduced.***
3. **Dispense now the unitary actor assumption and let Washington alone choose whether the colonies go to war against the crown. Additionally, suppose now that Washington get different shares of the benefits minus cost under peace and war. *Then, there could be some circumstances under which war is preferable.***

Proofs

1. Washington decides on war and peace on behalf of his people. Thus, from the colonial side what is relevant is Washington's best response:

$$\begin{aligned} E[U_W(peace, peace)] &\geq E[U_W(war, peace)] \\ b_p \times x_C &\geq p_C \times [b_w \times (R - \alpha c)] \end{aligned}$$

not the Colonial best response:

$$\begin{aligned} E[U_C(peace, peace)] &\geq E[U_C(war, peace)] \\ x_C &\geq p_C \times (R - c) \end{aligned}$$

The unitary actor assumption, however, implies that $b_p = b_w = 1$ and α=1. That is, Washington considers the full costs and benefits of war to his people, so that his best response is peace if:

$$\begin{aligned} E[U_W(peace, peace)] &\geq E[U_W(war, peace)] \\ x_C &\geq p_C \times (R - 1c) \\ x_C &\geq 40 \end{aligned}$$

This is the same result as in the earlier example. If the adversaries are evenly matched, the violent gamble is worth no more than $40 in expectation—a 50% shot at the pie, minus the costs from fighting. For a unitary actor (the ultimate benevolent dictator) any peaceful split that offers more than 40% of the pie is a better deal than war. For both sides, any split $x_i \in [40, 60]$ implies the strategy profile (peace, peace) is a Nash equilibrium.

2. Dispensing with the assumption that $\alpha = 1$, Washington's best response is:

$$\begin{aligned} E[U_W(peace, peace)] &\geq E[U_W(war, peace)] \\ x_C &\geq p_C \times (R - \alpha c) \\ x_C &\geq 0.5 \times (100 - \alpha 20) \\ x_C &\geq 50 - 10\alpha \end{aligned}$$

Then the new bargaining range (of distributions of Washington) is $[50 - 10\alpha, 60]$ which is smaller than the previous one $[40, 60]$. We can see that on the limit when $\alpha = 0$, the bargaining range is $[50, 60]$.

In this instance, by nominating a hawkish leader who ignores the costs of war, the Colonists could be said to have improved the bargaining range in their favor while still expecting peace.

3. Dispensing with the assumption that $b_p = b_w = 1$, peace is Washington's best response if:

$$\begin{aligned} E[U_W(peace, peace)] &\geq E[U_W(war, peace)] \\ b_p \times x_C &\geq p_C \times [b_w \times (R - \alpha c)] \\ b_p \times x_C &\geq 0.5 \times [b_w \times (100 - 20\alpha)] \\ x_C &\geq (50 - 10\alpha)\frac{b_w}{b_p} \end{aligned}$$

The British Crown is a unitary actor in this example. Thus the largest share of the pie that King George will willingly offer Washington and the colonists in peace is \$60. Suppose that Washington internalizes all costs of war ($\alpha = 1$). Then his best response is peace only if:

$$\begin{aligned} 60 &\geq 40\frac{b_w}{b_p} \\ \frac{3}{2} &\geq \frac{b_w}{b_p} \end{aligned}$$

This means that the share that Washington receives after war is at least 1.5 times the share he receives under peace to make peace unsustainable (less if he does not internalize all the costs of war).

3 Chapter 3: When Violence is Valued

Campesinos and Elites

Setup

Suppose the pie is control of El Salvador's vast coffee haciendas. Instead of gangs or British colonials, however, our two sides are now peasants and elites. The dispossessed campesinos have organized themselves for the first time. They're a threat to the oligarchic order, with even odds of victory. The elites have a choice. They can concede to peasant power, break up some of the biggest estates into cooperatives, but still hold on to half the land. Or they can fight and try to keep it all. Victory would cement their system of haciendas and oppression, minus the costs of war. The costs of war open a bargaining range of \$20 wide. This should be ample room for land reforms and representation to keep the campesinos from revolt.

Once upon a time, it seemed like serfdom was the natural order of things and campesinos were treated no better than animals. Hence, now introduce a violent value: righteous outrage. While anger sweeps peasants across the country. The emotional rewards offsets a fraction of the total costs of war. Still there is a 50% probability of victory for each group and each group knows the payoffs and the probability of victory. Initially each group has 50% of the haciendas.

Notation

The notation is the same as above, but now

- Subscript i denotes Campesinos (C) and Elites (E).
- v_i represents any benefits of righteous vengeance that either side receives

Claims in the book

1. ***Even though the bargaining range has reduced by half, peace will still be preferred over war.***

Proof

1. Now we will find the range of transfers that make for each group peace a best response, given that the other group choose peace. In the case of the Campesinos we need that:

$$\begin{aligned} E\left[U_C(\text{peace, peace})\right] &\geq E\left[U_C(war, peace)\right] \\ x_C &\geq p_C \times (R - c + v) \\ x_C &\geq 0.5 \times (80 + v) \\ x_C &\geq 40 + \frac{v}{2} \end{aligned}$$

In turn, for the Elites we need that:

$$\begin{aligned} E\left[U_E(peace, peace)\right] &\geq E\left[U_E(war, peace)\right] \\ x_P &\geq 0.5 \times 80 \\ x_P &\geq 40 \end{aligned}$$

Then as long as $v \leq 40$, there is a bargaining range defined by $x_C \in [40 + \frac{v}{2}, 60]$, and for any x in that range (peace,peace) is a Nash equilibrium. Notice then that the the campesinos have in some sense benefited from the new bargaining range, for their vengeance has closed off a set of unfavorable but peaceful bargains.

Finally, if $v = 20$, then we get the figure in chapter 2 where $x \in [0, 10]$.

4 Chapter 4: Uncertainty and information problems

4.1 Uncertainty as noise (Different prior beliefs)

Notation

Subscript i denotes each of the two rival groups: Vice Lords (V) and the Black P. Stones (B), and all other notation is the same as before.

Setup (Vice Lords and Black P. Stones)

Let's consider what happens when two gangs have different predictions about who will win a conflict. Take the case of the Vice Lords who believed they were well matched against the Stones. Let's imagine the Lords were looking at the familiar-looking \$100 pie from the previous sections, where the bargaining range is between \$40 and \$60. But suppose the Stones saw little bitty Nap Dog (the Vice Lord's leader), the untested 17-year old chief, and figured that times have changed. Suppose they thought they'll win 80 percent of the time if it comes to a fight. Of course, as usual, the total cost of war is \$20. We will assume that each group have 50% of the pie initially.

Claims in the book

1. ***Under these circumstances the bargaining range disappears and peace is no longer sustainable.***

Proof

The Vice Lords haven't changed their expected probabilities of victory under Nap Dog, and so their best response is peace if:

$$\begin{aligned} E\left[U_V(peace, peace)\right] &\geq E\left[U_V(war, peace)\right] \\ x_V &\geq p_V \times R - c \\ x_V &\geq 0.5 \times 80 \\ x_V &\geq 40 \end{aligned} \tag{1}$$

The Stones have a different assessment of probabilities, and prefer peace if:

$$\begin{aligned} E\left[U_B(peace, peace)\right] &\geq E\left[U_B(war, peace)\right] \\ x_B &\geq p_B \times R - c \\ x_B &\geq 0.8 \times 80 \\ x_B &\geq 64 \end{aligned} \tag{2}$$

This means that the Black P. Stones will never offer the Vice Lords more than \$36. Consequently, peace is no longer sustainable as there is any value of x that makes (peace,peace) a Nash equilibrium.

Previously, peace was a Nash equilibrium for the simple reason that both sides held the same prior beliefs. When we relax this "common priors" assumption, fighting becomes a sort of learning-by-doing.

This is an extreme example, with no overlap at all. If we picked a less extreme case, where the two rivals' priors aren't quite so far apart, then the bargaining ranges could intersect. But that range is narrower than if they'd started with the same reading of the situation.

Note that, a fuller model might account for the fact that the Stones are less confident in their new assessment, and the conclusions would be more nuanced. But this simple model gives a sense of what differing beliefs can do.

4.2 Private information and bluffing

Setup

Now, the uncertainty takes a different form: Are the Vice Lords weak or strong? Suppose the old men at the top of the Vice Lords know a bitter truth—they are sapped and outmatched at Horner. They figure they have a three-quarters chance of losing a war against the Stones. After all, their local leader is 17. And the Stones just raided their building with Uzis!

But there is hope. That's because the Stones aren't certain of the truth. The Stones assign some chance that the Lords have grown weak, and some chance they're just as strong as they originally were (where they were as strong as the Stones). Only the Lords know the reality—they have what's called "private information". The Stones know that both scenarios are possible. They also know that the Lords have an incentive to bluff, and they regard every signal suspiciously.

As usual, assume the total cost of war is \$20 (the shrunken pie worth \$80) and the total pie is worth \$100. Also, assume that if a bluff is called, it results in a war. Otherwise, the result is peace.

Notation

Subscript i denotes each of the two rival groups: Vice Lords (V) and the Black P. Stones (B), and all other notation is the same as before.

Previous games where simultaneous as players choose their action at the same time. In this case players play at different times and later players can see some of the previous actions. There is order in how actions are executed and players can choose their actions when they have to make a decision. This is known as an extensive form game. A sequence of actions in this game will be called a history.

The game will follow the following sequence. In the starting point "nature" will choose the type of the Vice Lords, such as with probability τ they are strong, and with probability $(1-\tau)$ they are weak. Then, the Vice Lords, who know their type, will choose whether to threat or not the Stones. Then, the Stones, who know the previous action but not the type of the Vice Lords, will choose to offer a high or a low offer (40 or 16 respectively). Finally, the Vice Lords will decide whether to accept the distribution or go to war. We will assume that the strong type only accepts a high offer, and that the weak type accept any offer so we will consider the last step directly in the payoff rather than an extra action.

We will focus on pure strategies where each group chooses an action on each of it's information sets. Information sets are sets of histories in the game such as any history within the set is indistinguishable from the groups perspective, and as this is a sequential game with imperfect information some information sets won't be singletons. In this game we the Vice Lords have two information sets: $\{\{strong\}, \{weak\}\}$, as each information set contains a singleton they have full information. In the case of the Stones they have two information sets as well $\{\{(strong, Threat), (weak, Threat)\}, \{(strong, NoThreat), (weak, NoThreat)\}\}$. In the case of the Stones they don't have perfect information because when they see each action of the Vice Lords they don't know if nature chose before strong or weak type.

We will denote s_i a pure strategy of group i, that will specify an action for each information set. In the case of the Vice Lords they have 2 possible actions ($Threat, NoThreat$), but as they could have 2 types they have 4 possible strategies: $\{(Threat, Threat), (Threat, NoThreat), (NoThreat, Threat), (NoThreat, NoThreat)\}$ were the first action denote their plan when they are of the strong type, and the second one when they are of the weak type. In the case of the Stones they have two possible actions: to offer a high offer (40) or a low offer (16). As they face 2 information sets they have 4 strategies as well: $\{(High, High), (High, Low),$ $(Low, High), (Low, Low)\}$, where the first action denote the plan when the Vice Lords choose Threat, and the second one when the Vice Lords choose No Threat.

As the Stones face two information sets, where they don't know which of the 2 possible histories happened, they will form beliefs. We will denote with $\mu(.)$ the probability the Stones assign that the Vice Lords are strong given the action of the Vice Lords.

As an illustrative example we will assume that when the Lords are of the strong type thread is a dominant strategy, but the weak type utility is equivalent of the expected value of the share they get minus 1 if they threat.

$E[U_i(s_i, s_j)]$ will denote the (expected) payoff of group i of choosing strategy s_i given that the other group chooses strategy s_j.

The equilibrium concept in this sequential game of imperfect information is (weak) Perfect Bayesian Equilibrium (PBE). It consist in (s, μ) representing a strategy profile and a behavioral strategy, which assign probabilities of action on each information set, and belief system that satisfies sequential rationality and consistency of beliefs with strategies. Sequential rationality means that each group choose the best available action given it's belief and the other group strategies. Consistency of beliefs with strategies means that for every information set reached with non-zero probability, given strategy profile s, μ is restricted to follows Bayes' rule to assigns probabilities to each node in the information set.

Claims in the book

1. **Assume the Stones belief that the Lords are strong is given by μ, which is a distribution over the possible types assigned to the Lords (states of the world). Further assume that (i) the actions for the Lords are to make a threat (T) and to not make a threat (NT), the actions for the Stones are to offer a low amount of money (L) and to offer a high amount of money (H), and (ii) if the Lords are strong, they will always prefer to make a threat (T).** ***Then, there exist then an equilibrium where peace is sustained and another one where war can occur.***

Proof

One possibility is that both types of Lords (weak and strong) choose to threat. If the strong and the weak type of the Lords choose to threat then the Stones would choose to give a big share if the likelihood of the Lords being strong is high enough. If the Stones give a low share if the Lords play no threat, even the weak Lords would choose threat over no threat. Then even if the Lords are weak they could choose to threat the Stones always.[3]

We will discuss in more detail the case of the semi-separating equilibrium. This arise when the strong type always choose to threat and the weak type choose to threat with probability $p \in (0, 1)$, and the Stones choose to give a low share with probability $q \in (0, 1)$.

First, we need to calculate the beliefs of the Stones about the types given the action they see. Let's calculate the belief of the Stones that the Vice Lords are strong given seeing the action Threat. As this action happens with positive probability it is restrained to follow Bayes' rule. This will make that the probability of

$$
\begin{aligned}
\mu(Strong|Threat) &= \frac{P(Threat|Strong) \times P(Strong)}{P(Threat)} \\
&= \frac{P(Threat|Strong) \times P(Strong)}{P(Threat|Strong) \times P(Strong) + P(Threat|Weak) \times P(Weak)} \\
&= \frac{\tau}{\tau + (1-\tau)p} \qquad (3)
\end{aligned}
$$

Then they believe that are strong with probability $\frac{\tau}{\tau+(1-\tau)p}$ given that Vice Lords choose Threat. Vice Lords could choose choose No Threat with positive probability and it will happen only if they are of weak type. Then, $\mu(Strong|NoThreat) = 0$[4] Given the beliefs of the Stones the payoff of each action given that

[3] More formally this is called a pooling equilibrium. As only the information set after player 1 choosing threat is achieved with positive probability, player 2 would need to use Bayes rule only there. We would need to have beliefs such as for player 2 is optimal to give big share after seeing threat but low after seeing no threat. This would make both types to choose threat.

[4] We need to follow Bayes rule again:

$$
\begin{aligned}
\mu(Strong|NoThreat) &= \frac{P(NoThreat|Strong) \times P(Strong)}{P(NoThreat)} \\
&= \frac{P(NoThreat|Strong) \times P(Strong)}{P(NoThreat|Strong) \times P(Strong) + P(NoThreat|Weak) \times P(Weak)} \\
&= 0 \qquad (4)
\end{aligned}
$$

the Vice Lords choose Threat is:

$$\begin{aligned}
E[U_B(High,T)] &= \mu(Strong|Threat)\times U_B(High,T|Strong)+\mu(Weak|Threat)\times U_B(High,T|Weak) \\
E[U_B(High,T)] &= \tau\times 60+(1-\tau)\times 60=60 \qquad (5)\\
E[U_B(Low,T)] &= \mu(Strong|Threat)\times U_B(High,T|Strong)+\mu(Weak|Threat)\times U_B(High,T|Weak) \\
E[U_B(Low,T)] &= \frac{\tau}{\tau+(1-\tau)p}\times 40+(1-\frac{\tau}{\tau+(1-\tau)p})\times 84 \\
E[U_B(Low,T)] &= \frac{40\tau+84(1-\tau)p}{\tau+(1-\tau)p} \qquad (6)
\end{aligned}$$

For obtaining a semi-separating equilibrium we need that the Stones are indifferent between choosing any action when the Vice Lords choose Threat. Then we need that:

$$\begin{aligned}
E[U_S(High,T)] &= E[U_S(Low,T)] \\
60 &= \frac{40\tau+84(1-\tau)p}{\tau+(1-\tau)p} \\
60\tau+60(1-\tau)p) &= 40\tau+84(1-\tau)p \\
20\tau &= +24(1-\tau)p \\
p &= \frac{5\tau}{6(1-\tau)} \qquad (7)
\end{aligned}$$

Then a necessary condition for having a semi-separating equilibrium is:

$$\begin{aligned}
p=\frac{5\tau}{6(1-\tau)} &\leq 1 \\
5\tau &\leq 6(1-\tau) \\
\tau &\leq \frac{6}{11} \qquad (8)
\end{aligned}$$

In the case of no threat the Stones believe that with probability 1 the Vice Lords are of the weak type so they choose the low offer.

Now we need to check that any of the two types have any incentive to deviate. In the case of the strong type we already know that they prefer to choose threat independent of the other player action so they would choose threat. In the case of the weak type we need that they are indifferent between both actions so they could choose a mixed strategy. They have the following payoffs:

$$\begin{aligned}
U_V(T,q\times High+(1-q)\times Low|Weak) &= q\times(40-1)+(1-q)\times 15 \\
U_V(T,q\times High+(1-q)\times Low|Weak) &= 15+24q \qquad (9)\\
U_V(NT,Low|Weak) &= 16 \qquad (10)\\
& \qquad (11)
\end{aligned}$$

Then the condition for indifference is:

$$\begin{aligned}
U_V(T,q\times High+(1-q)\times Low|Weak) &= U_V(NT,Low|Weak) \\
15+24q &= 16 \\
q &= \frac{1}{24} \qquad (12)
\end{aligned}$$

Then there is a semi-separating equilibrium with positive probability of war as with probability $q = 1/24$ the Stones offer low offer which the strong type won't accept and would declare war. This means that there is a positive change of bluff and a positive probability of being called.

5 Chapter 5: Commitment problems

Sparta and Athens

Setup

Turning the classical Greek world into a now familiar \$100 pie, let's suppose that at the outset of the 5th century BC (around the time the Persians were expelled from mainland Greece) Sparta and its Peloponnesian League could win a war against Athens and its allies 75% of the time.

Athens then begins its ascent and at the same time, Sparta also suffered setbacks. As a result, by the middle of the 5th century, suppose that Sparta could foresee a day when the balance of power would be more even where both sides have 50% chance of winning a war and each having 50% of the pie. A crucial detail is that this shift from a 75-25 to a 50-50 match in military power hasn't come about, yet. The rebalancing could be averted if Sparta goes to war and wins.

Let's imagine there are two periods: today and the future. Sparta and Athens are not just bargaining over today's \$100 pie, the contest is also for \$100 in the future, and there is no discounting. War would shrink today's pie as well as the future one, by \$20 each year. That means war destroys \$40 in total. So, today and the future's war-damaged pies are worth \$160 together.

Notation

Subscript i denotes each of the two rival groups: Sparta (S) and Athens (A). j is just the notation assigned to the rival of i. We will denote s_i a pure strategy of group i on the extensive game. As it is a full information game it assigns an action to each turn of the player. Each action could be either war or bargain.

The game has the following form: in the first stage Athens offer a distribution x_1 to Sparta, in the second stage Sparta either accepts the offer or go to war, in the third stage Athens offer a new distribution to Sparta, in the fourth and final stage Sparta accepts or goes to war.

Then Athens strategies correspond to all pairs (x_1, x_2) were $x_t \in [0, 100], \text{for} t = 1, 2$, that represent the offered allocation to Sparta. Sparta possible pure strategies are $\{(Accept, Accept), (Accept, War), (War, Accept), (War, War)\}$, where each pair represent the action of period 1 and 2 respectively.

$E[U_i(s_i, s_j)]$ will denote the (expected) payoff of group i of choosing strategy s_i given that the other player chooses strategy s_j. It measures the expected value they will have for a particular strategy profile.

Finally, a Subgame Perfect Nash Equilibrium (SPNE) is of the form: $\{(\cdot, \cdot); (\cdot, \cdot)\}$. Where the first entry of $\{\}$ is given by the strategy of Athens at period 0 and at period 2, respectively, and the second entry of $\{\}$ is given by the strategy of Sparta at period 1.

Claims in the book

1. **Sparta needs a promise of at least \$120 not to invade. That demand looks feasible. Knowing it can get at least \$40 in the future (maybe as much as \$60) all Sparta needs today is to get at least \$80. That's a high price for Athens today—it means wagonloads of tribute, a whole colony, or other exorbitant concessions. But peace is conceivable. There's no fundamental commitment problem here, despite a huge expected shift in relative strength. *Sparta will believe that Athens will do this transfer and peace is going to be the equilibrium.***

2. **Let's study the following illustrative example. The chances of winning for Sparta in period 1 are the same as before, but now considering a power shift in period 2 where now Athens has a probability of winning of** $13/16$. ***Now, Sparta will not believe that Athens will do the transfer that guarantees peace, so war is going to be the equilibrium.***

Proof

1. We will proceed to solve the game by backward induction. If in period 1 there is peace, in the last node Sparta has two available actions: $\{War, Peace\}$. Its best response would be peace under the following condition:

$$\begin{aligned} U_S(Peace_2) &\geq U_S(War_2) \\ x_1 + x_2 &\geq x_1 + 0.5 \times (100 - 20) + 0.5 \times 0 \\ x_2 &\geq 40 \end{aligned} \tag{13}$$

So Athens need to offer at least 40 in the second period so peace is a best response to Sparta. In the previous stage Athens could choose any $x_2 \in [0, 100]$. If Athens choose $x_2 \geq 40$ Sparta would choose peace so it's payoff would be:

$$U_A(x_2 \geq 40) = (100 - x_1) + (100 - x_2) = 200 - x_1 - x_2 \tag{14}$$

If Athens increase x_2 it will decrease it's payoff (we can see by taking the derivative of the previous expression), so for that segment the payoff is highest at $x_2 = 40$:

$$U_A(x_2 = 40) = (100 - x_1) + (100 - x_2) = 160 - x_1 \tag{15}$$

If Athens decrease x_2 Sparta would choose war, leading to the following payoff to Athens:

$$U_A(x_2 < 40) = (100 - x_1) + 0.5 \times (100 - 20) + 0.5 \times 0 = 140 - x_1 \tag{16}$$

So Athens would choose $x_2 = 40$ in that node. Sparta would consider the previous results when choosing whether to accept or not the first proposal of Athens. The best response of Sparta is to accept the first offer under the following condition:

$$\begin{aligned} U_S(Peace_1) &\geq U_S(War_1) \\ x_1 + 40 &\geq 0.75 \times (100 - 20 + 100 - 20) + 0.25 \times 0 \\ x_1 &\geq 80 \end{aligned} \tag{17}$$

So Sparta will accept peace as long as $x_1 \geq 80$. Athens, by an analogous comparison, would choose $x_1 = 80$, as increasing x_1 strictly decrease its payoff and reducing it leads to war in period 1 with payoff 45. Then we have a SPNE with peace consisting in the strategy profile $((80, 40), (Peace, Peace))$. There is no fundamental commitment problem here, despite huge expected shift in power.

2. We will again proceed to solve the game by backward induction. If in period 1 there is peace, in the last node Sparta have two available actions: $\{War, Peace\}$. Its best response would be peace under the following condition:

$$\begin{aligned} U_S(Peace_2) &\geq U_S(War_2) \\ x_1 + x_2 &\geq x_1 + 0.1875 \times (100 - 20) + 0.8125 \times 0 \\ x_2 &\geq 15 \end{aligned} \tag{18}$$

So Athens need to offer at least 15 in the second period so peace is a best response to Sparta. In the previous stage Athens could choose any $x_2 \in [0, 100]$. If Athens choose $x_2 \geq 15$ Sparta would choose peace so it's payoff would be:

$$U_A(x_2 \geq 15) = (100 - x_1) + (100 - x_2) = 200 - x_1 - x_2 \quad (19)$$

If Athens increase x_2 it will decrease it's payoff (we can see by taking the derivative of the previous expression), so in that segment Athens would choose $x_2 = 15$ with payoff: $185 - x_1$. If Athens decrease x_2 Sparta would choose war, leading to the following payoff to Athens:

$$U_A(x_2) = (100 - x_1) + 0.8125 \times (100 - 20) + 0.1875 \times 0 = 165 - x_1 \quad (20)$$

So Athens would choose $x_2 = 15$ in that node. Sparta would consider the previous results when choosing whether to accept or not the first proposal of Athens. In the previous stage Sparta would make the following comparison:

$$\begin{aligned} U_S(Peace_1) &< U_S(War_1) \\ x_1 + 15 &< 0.75 \times (100 - 20 + 100 - 20) + 0.25 \times 0 \\ x_1 &< 105 \end{aligned} \quad (21)$$

As Athens can only transfer 100 at most in period one, the commitment problem leads to war.

6 Chapter 6: Integration

Hindus and Muslims

Setup

Let's imagine two groups (Hindus and Muslims) that are competing for a pie valued in \$100, but that are economically intertwined. We can think of lending networks, business relations, or general complementarities. Let's assume that war costs 20 in destruction but that war also affect by another way depending on the level of integration (ι). In particular, for a group that goes to war and result victorious, their valuation of the pie would be given by $\frac{100-20}{1+\iota}$. We will assume that initially each group has half of the pie.

Notation

Subscript i denotes each of the two rival groups: Hindus (H) and Muslims (M). j is just the notation assigned to the rival of i. We will denote s_i a pure strategy of group i, which can be either war or peace.[5] $E[U_i(s_i, s_j)]$ will denote the (expected) payoff of group i of choosing strategy s_i given that the other player chooses strategy s_j. It measures the expected value they will have for a particular strategy profile. $x \in [0, 50]$ is the amount of the transfer that Hindus have to give to Muslims. As originally Hindus have 50% of the pie that's the maximum they can give. Finally, $\iota \geq 0$ denotes the level of integration between groups.

Claims in the book

Introducing a positive level of integration between groups makes the bargaining range wider.

Proof

Before the wedge was \$20 wide because one side expected to win a damaged pie. They didn't care that their rival lost everything. But suppose the size of the pie depends on both groups cooperating. If you

[5] s_i could also be a probability distribution over war and peace (e.g. play with 60% of probability war and 40% of probability peace) known as mixed strategies. We will focus on equilibrium in pure strategies so we won't consider this kind of strategies.

eliminate the other side, you undermine the community's capital stocks, trade networks, and productivity. The post-war pie is actually smaller. We will show how integration affects the bargaining range that sustain a (peace,peace) Nash equilibrium. Peace will be the best response of Hindus, if Muslims choose peace, if:

$$\begin{aligned} E[U_H(peace, peace)] &\geq E[U_H(war, peace)] \\ 50 - x &\geq 0.5 \times \frac{100-20}{1+\iota} + 0.5 \times 0 \\ 50 - \frac{40}{1+\iota} &\geq x \end{aligned} \tag{22}$$

Analogously for Muslims we obtain that:

$$\begin{aligned} E[U_M(peace, peace)] &\geq E[U_M(war, peace)] \\ 50 + x &\geq 0.5 \times \frac{100-20}{1+\iota} + 0.5 \times 0 \\ x &\geq \frac{40}{1+\iota} - 50 \end{aligned} \tag{23}$$

Thus, the bargaining range is $x \in [\frac{40}{1+\iota} - 50, 50 - \frac{40}{1+\iota}]$. We can see that without integration ($\iota = 0$) we have the initial bargaining range ($x \in [-10, 10]$), but that the larger the integration (ι) between the groups is, the larger is going to be the bargaining range. In the limit, if both sides are extremely integrated such that $\iota \to \infty$ the bargaining range becomes $x \in [-50, 50]$. In other words, as the level of integration is maximum between both groups, the bargaining range becomes the whole pie!